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Pseudospectral Methods for Infinite-Horizon Optimal Control Problems

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A central computational issue in solving infinite-horizon nonlinear optimal control problems is the treatment of the horizon. In this paper, we directly address this issue by a domain transformation technique that maps the infinite horizon to a finite horizon. The transformed finite horizon serves as the computational domain for an application of pseudospectral methods. Although any pseudospectral method may be used, we focus on the Legendre pseudospectral method. It is shown that the proper class of Legendre pseudospectral methods to solve infinite-horizon problems are the Radau-based methods with weighted interpolants. This is in sharp contrast to the unweighted pseudospectral techniques for optimal control. The Legendre–Gauss–Radau pseudospectral method is thus developed to solve nonlinear constrained optimal control problems. An application of the covector mapping principle for the Legendre–Gauss–Radau pseudospectral method generates a covector mapping theorem that provides an efficient approach for the verification and validation of the extremality of the computed solution. Several example problems are solved to illustrate the ideas.

I. Introduction

OVER the last few years, it has become possible to routinely solve many practical finite-horizon optimal control problems. Practical problems are typically nonlinear and constrained in both the state and control variables. A recent example of generating routine solutions to practical problems is the flight implementation of pseudospectral solutions to an optimal "zero-propellant maneuver" of the International Space Station [1,2]. Furthermore, a fairly large class of finite-horizon problems can also be solved in real time. A number of issues related to this advancement are discussed in [3]. Real-time computation of optimal controls implies optimal feedback guidance; hence, a number of control problems that are not readily addressable by traditional methods can now be solved in a unified manner [4]. Despite such advancements, obtaining feedback controls for stabilizing a nonlinear system poses significant challenges [5]. Among many, one of the reasons for the continuing challenge is the treatment of the infinite horizon in a nonlinear optimal control problem [6]. A well-known strategy to address this issue is the receding-horizon technique, which forms the basis of model-predictive methods for control. Even though finite-horizon problems are solved in real time, in this approach, the finite horizon is receded indefinitely as a means to approximate the infinite horizon. As a result of the mismatch between the computational horizon and the infinite horizon, a large number of new problems emerge. Consequently, different flavors of model-predictive control techniques have been proposed to address these problems. These issues are surveyed in [6] and the references contained therein.

In this paper, we propose an altogether different approach for solving infinite-horizon, nonlinear, constrained optimal control problems. Our approach follows [7], in which we presented some initial ideas; here, we further the concepts along several different directions. First, the infinite horizon is mapped to a finite horizon by a domain transformation technique that preserves the notion of the infinite horizon. Second, a pseudospectral (PS) method is applied to discretize the problem. The discretized problem generates a sequence of approximate problems parameterized by the size of the grid. In the third and final step, each problem in this sequence is solved by a globally convergent sequential quadratic programming (SQP) technique [8]. Note that global convergence does not mean global optimality [9]. Among many, this is one reason why it is important to select a correct sequence of problems together with the SQP subsequence to design a proper algorithm for solving optimal control problems [10]. Such an algorithm for PS methods is called a spectral algorithm [11], which forms the core of PS methods for control. For the purpose of brevity, we do not discuss all aspects of our proposed approaches because many of these concepts have been extensively discussed elsewhere. For example, a complete description of the Legendre–Gauss–Lobatto PS method is described in [12,13]; the spectral algorithm is described in [11]; and a convergence theorem is developed in [10]. *Consequently, in this paper, we limit the scope of the problem to a complete discussion of only the open-loop issues pertaining to the treatment of the infinite horizon*. Stability of the closed-loop system is discussed in [14]. An overview of an implementation of the closed-loop control is discussed in [15]. In this context, this paper provides the foundations for solving infinite-horizon optimal control problems. Part of this foundation is the development of a Radau-based PS method. It turns out that the well-known Lobatto family of PS methods is inappropriate for the treatment of the infinite horizon as a result of singularities in the domain transformation technique. A detailed discussion of these issues is included in this paper.

As a matter of context, we note that our proposed approach is *not* an application of PS discretization applied to receding-horizon techniques. This approach has been discussed elsewhere [16–20] and offers well-established benefits of PS methods; however, as alluded to earlier, the fundamental problems and issues [6] of receding-horizon techniques arising from the finiteness of solving the finite-horizon optimal control problem do not disappear by choosing alternative discretizations. Thus, the proposed approach in this paper provides a foundation for a new approach to solving infinite-horizon optimal control problems.

Two example problems are solved to illustrate various elements of the main points of the proposed ideas: one is a standard linear-quadratic regulator (LQR) problem in which the solutions from our method are compared favorably against the solutions from the Riccati approach. The second example is a nonlinear control problem of stabilizing NPSAT1, an experimental spacecraft designed and built at the Naval Postgraduate School and scheduled to be launched

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in fall 2009. Additional example problems have been solved [7,14] but are not discussed in this paper for the purpose of brevity.

II. Problem Formulation

We consider the following infinite-horizon, constrained, nonlinear optimal control problem:

$$({}^{\infty}B)\begin{cases}\text{Minimize} & J[\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)]=\int_0^{\infty}F(\boldsymbol{x}(t),\boldsymbol{u}(t))dt\\ \text{Subject to} & \dot{\boldsymbol{x}}(t)=\mathbf{f}(\boldsymbol{x}(t),\boldsymbol{u}(t))\\ & \boldsymbol{x}(0)=\boldsymbol{x}^0\\ & \mathbf{h}(\boldsymbol{x}(t),\boldsymbol{u}(t))\le 0\end{cases}$$

where $[0,\infty)\ni t\mapsto\{\boldsymbol{x}\in\mathbb{R}^{N_x},\boldsymbol{u}\in\mathbb{R}^{N_u}\}$ is the state-control function pair; $F\colon \mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}$, $\mathbf{f}\colon \mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}^{N_x}$, and $\mathbf{h}\colon \mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}^{N_h}$ are continuously differentiable functions; and $\boldsymbol{x}^0$ is a given value of the state at the initial time $t=0$. The symbol N denotes the vector dimension (i.e., number of variables) of the subscripted quantities. Thus, it is clear that we are dealing with a fairly general optimal control problem for which the problem formulation is motivated by stabilizing nonlinear control systems that frequently describe high-performance aerospace vehicles. We assume that a solution exists for problem ${}^{\infty}B$ in some appropriate Sobolev space [21] and that the limits

$$\lim_{t\to\infty}\boldsymbol{x}(t)=\boldsymbol{x}(\infty)\quad\text{and}\quad\lim_{t\to\infty}\boldsymbol{u}(t)=\boldsymbol{u}(\infty)$$

exist. Then, under appropriate conditions [22], the Pontryagin minimum principle holds for problem ${}^{\infty}B$. That is, if $\{\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)\}$ is a solution to problem ${}^{\infty}B$, then there exist multipliers (duals or covectors) that satisfy the adjoint equation, the Hamiltonian minimization condition, and the transversality conditions. This dualization may be cast in terms of a boundary-value problem, denoted as problem ${}^{\infty}B^{\lambda}$, and is defined as

$$({}^{\infty}B^{\lambda})\begin{cases}\text{Find} & [\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),\boldsymbol{\lambda}(\cdot),\boldsymbol{\mu}(\cdot)]\\ \text{Such that} & \dot{\boldsymbol{x}}(t)=\mathbf{f}(\boldsymbol{x}(t),\boldsymbol{u}(t))\\ & \boldsymbol{x}(0)=\boldsymbol{x}^0\\ & \mathbf{h}(\boldsymbol{x}(t),\boldsymbol{u}(t))\le 0\\ & \frac{\partial\bar{H}[t]}{\partial\boldsymbol{u}}=\mathbf{0}\\ & \boldsymbol{\mu}(t)^T\mathbf{h}(\boldsymbol{x}(t),\boldsymbol{u}(t))=\mathbf{0}\\ & \boldsymbol{\mu}(t)\ge\mathbf{0}\\ & \dot{\boldsymbol{\lambda}}(t)=-\frac{\partial\bar{H}[t]}{\partial\boldsymbol{x}}\\ & \boldsymbol{\lambda}(\infty)=\mathbf{0}\end{cases}$$

where $\bar{H}\colon \mathbb{R}_+^{N_h}\times\mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}$ is the D form [23] of the Hamiltonian–Lagrangian

$$\bar{H}(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{x},\boldsymbol{u}):=H(\boldsymbol{\lambda},\boldsymbol{x},\boldsymbol{u})+\boldsymbol{\mu}^T\mathbf{h}(\boldsymbol{x},\boldsymbol{u})\tag{1}$$

$H\colon \mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\to\mathbb{R}$ is the Pontryagin Hamiltonian

$$H(\boldsymbol{\lambda},\boldsymbol{x},\boldsymbol{u}):=F(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{\lambda}^T\mathbf{f}(\boldsymbol{x},\boldsymbol{u})\tag{2}$$

and $\bar{H}[t]$ is a shorthand notation [21] for $\bar{H}(\boldsymbol{\mu}(t),\boldsymbol{\lambda}(t),\boldsymbol{x}(t),\boldsymbol{u}(t))$. As in the case of the states and controls, we use the notation

$$\boldsymbol{\lambda}(\infty)=\lim_{t\to\infty}\boldsymbol{\lambda}(t)$$

for the limit and assume that it exists.

III. Domain Transformation Technique

As noted earlier, the semi-infinite domain $[0,\infty)$ poses a number of problems. That is, it is necessary to properly account for the infiniteness of the time domain. In [24], we proposed a Laguerre PS method as a means to address this problem head-on. Motivated by domain transformation techniques prevalent in PS methods, and our previous success in designing an adaptive PS method for control [25], we let

$$t=t(\tau):=\frac{1+\tau}{1-\tau}\Leftrightarrow\tau=\frac{t-1}{t+1}\tag{3}$$

where $\tau\in[-1,1]$. Because we assumed that quantities exist in the limit as $t\to\infty$, it is clear that Eq. (3) maps the infinite domain to a finite domain. Using

$$\frac{dt}{d\tau}=\frac{2}{(1-\tau)^2}:=r(\tau)\tag{4}$$

we reformulate problem ${}^{\infty}B$ on the finite interval $[-1,1]$ as

$$(B)\begin{cases}\text{Minimize} & J[\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)]=\int_{-1}^{1}r(\tau)F(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))d\tau\\ \text{Subject to} & \frac{d\boldsymbol{x}(t(\tau))}{d\tau}=r(\tau)\mathbf{f}(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))\\ & \boldsymbol{x}(t(-1))=\boldsymbol{x}^0\\ & \mathbf{h}(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))\le 0\end{cases}$$

Evaluations at $\tau=1$ must be performed in the sense of the limit consistent with $t\to\infty$. In applying the Pontryagin minimum principle for problem B, we generate the following problem:

$$(B^{\lambda})\begin{cases}\text{Find} & [\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),\boldsymbol{\lambda}(\cdot),\boldsymbol{\mu}(\cdot)]\\ \text{Such that} & \frac{d\boldsymbol{x}(t(\tau))}{d\tau}=r(\tau)\mathbf{f}(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))\\ & \boldsymbol{x}(t(-1))=\boldsymbol{x}^0\\ & \mathbf{h}(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))\le 0\\ & \frac{\partial\bar{H}[\tau]}{\partial\boldsymbol{u}}=\mathbf{0}\\ & \boldsymbol{\mu}(t(\tau))^T\mathbf{h}(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))=\mathbf{0}\\ & \boldsymbol{\mu}(t(\tau))\ge\mathbf{0}\\ & \dot{\boldsymbol{\lambda}}(t(\tau))=-r(\tau)\frac{\partial\bar{H}[\tau]}{\partial\boldsymbol{x}}\\ & \boldsymbol{\lambda}(t(1))=\mathbf{0}\end{cases}$$

where evaluations at $t(1)$ are also to be understood in the sense of a limit.

The equivalence between problems ${}^{\infty}B$ and B require a set of technical assumptions that we implicitly assume. For the purposes of brevity, we do not list all these assumptions, but we note one of these to illustrate the point.

In problem ${}^{\infty}B$ it is necessary that the function $t\mapsto F(\boldsymbol{x}(t),\boldsymbol{u}(t))$ be in $L^1([0,\infty),\mathbb{R})$; however, this does not necessarily guarantee that $\tau\mapsto r(\tau)F(\boldsymbol{x}(t(\tau)),\boldsymbol{u}(t(\tau)))$ will be in $L^1([-1,1],\mathbb{R})$. Consequently, we implicitly assume that the transformed cost function is locally integrable.

IV. New Perspective on Pseudospectral Methods

Although recognizing that the domain transformation technique is quite independent of PS methods for solving optimal control problem, the rationale for transforming the domain from $[0,\infty)$ to $[-1,1]$ and not, say, [0, 1] is indeed motivated by pseudospectral methods. Two of the most widely used PS methods are the Legendre and Chebyshev pseudospectral methods. These methods rely on interpolation of the approximating polynomials for the states and costates on Gaussian quadrature points (such as Legendre- or Chebyshev–Gauss–Lobatto points). These points lie in the interval $[-1,1]$, but by using a time-domain transformation, any arbitrary finite interval can be mapped to this interval. Typically, a linear domain transformation is used; however, proper nonlinear domain transformation techniques can be used quite effectively to create an adaptive grid [25]. A simple extension of this idea is to map the semi-infinite domain to the finite time domain $[-1,1]$ and then use the appropriate quadrature nodes for the interpolation polynomials. Equation (3) is in fact the rational map commonly used [26,27] in pseudospectral methods for domain transformation. Other appropriate domain transformation techniques may be used as well, and as will be apparent in the sections to follow, our methods carry over quite trivially.

A. Overview

In PS methods, the functions are approximated by Lagrange interpolating polynomials of order N in which interpolation occurs at

Gaussian quadrature points that lie in the interval $[-1, 1]$. There are three different families of Gauss quadrature points known as Gauss, Gauss–Radau, and Gauss–Lobatto [27,28]. What distinguishes these nodes is the choice of zeros of orthogonal polynomials such as Jacobi polynomials, in general, or Chebyshev or Legendre polynomials in most applications. Gauss quadrature uses zeros that are interior to the interval $[-1, 1]$. The Gauss–Radau zeros include one of the endpoints of the interval, usually the left-endpoint at $\tau = -1$. The Gauss–Lobatto points include both endpoints of the interval at $\tau = -1$, and $\tau = 1$. Proper choice of these nodes depends on the boundary-value problem that is being solved. In a general optimal control problem over a finite-horizon, explicit formulations of initial and final conditions are often required. This is why Lobatto points, which are fixed at both of the boundary points, are the most natural choice. On the other hand, for the infinite-horizon optimal control problems, one generally has explicit boundary conditions at the initial point, and the implicit assumption is that solutions at infinity tend to zero. Therefore, for these problems, it suffices to use nodes that are fixed at the left-hand node. Moreover, in our technique of mapping the semi-infinite horizon to a finite interval, the derivative of the function $r(\tau)$ has a singularity at $\tau = 1$, which corresponds to the right-hand endpoint at infinity. Therefore, for these problems, only the left-hand boundary point -1 is required; hence, the Radau points, which are fixed at the left-hand endpoint $\tau = -1$, are more appropriate than Lobatto points. It should also be noted that in an implementation of PS methods, the Gauss points are not suitable because their use poses subtle but serious issues in applying the boundary conditions [29]. Consequently, we choose Legendre–Gauss–Radau (LGR) points for discretization. These points τ_j $(j = 0, \cdots, N)$ are defined as fixed at the initial point $\tau_0 = -1$ and the rest of the nodes are defined as zeros of $L_N + L_{N+1}$, where L_N is the Legendre polynomial of degree N. For these points, which are distributed over $[-1, 1)$, evaluation at the right-hand point (which, for the mapped domain, corresponds to ∞) is at $\tau_N = 1 - \epsilon$, where the size of ϵ depends inversely on N; that is, $\epsilon \to 0$ as $N \to \infty$. Because evaluations at ∞ are to be understood in the sense of the limit, it is clear that the last Radau node provides precisely this condition.

B. Approximation of Functions, Derivatives, and Integrals

To introduce the basic ideas, we consider a generic function $\tau \mapsto y(\tau) \in \mathbb{R}$. This allows us to ignore much of the bookkeeping associated with vector-valued functions. Pseudospectral methods are based on approximating $y(\tau)$ by way of weighted interpolants of the form [27,30]

$$y(\tau) \approx y^N(\tau) = \sum_{j=0}^{N} \frac{\alpha(\tau)}{\alpha(\tau_j)} \phi_j(\tau) y_j \tag{5}$$

where α is a positive weight function, $\{\tau_j\}_{j=0}^N$ are Gaussian points, and $\{\phi_j(\tau)\}_{j=0}^N$ satisfy $\phi_j(\tau_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta. Hence, we have

$$y^N(\tau_k) = y_k$$

For the LGR method, we choose $\alpha(\tau) = 1$ for approximating the states and $\alpha(\tau) = 1 - \tau$ for approximating the costates. *The motivation for this choice of weight functions is based on generating a consistent set of new approximations and is further elaborated in Sec. V.*

We approximate

$$\int_{-1}^{1} y(\tau) d\tau$$

by the LGR quadrature formula:

$$\int_{-1}^{1} y(\tau) d\tau \approx \int_{-1}^{1} y^N(\tau) d\tau = \sum_{j=0}^{N} \hat{w}_j y_j \tag{6}$$

where $\{\hat{w}_j\}_{j=0}^N$ are the LGR weights defined as

$$\hat{w}_j := \int_{-1}^{1} \frac{\alpha(\tau)}{\alpha(\tau_j)} \phi_j(\tau) d\tau \tag{7}$$

It is straightforward to show that $\hat{w}_j$ is independent of the choice of α and is given explicitly by [31]

$$\hat{w}_j = \begin{cases} \frac{2}{(N+1)^2} & j = 0 \\ \frac{1}{(N+1)^2} \frac{1-\tau_j}{[L_N(\tau_j)]^2} & j \neq 0 \end{cases} \tag{8}$$

The derivative of $y(\tau)$ is obtained by differentiating Eq. (5):

$$\frac{dy(\tau)}{d\tau} \approx \frac{dy^N(\tau)}{d\tau} = \sum_{j=0}^{N} \frac{d}{d\tau} \left(\frac{\alpha(\tau)}{\alpha(\tau_j)} \phi_j(\tau) \right) y_j \tag{9}$$

Evaluating the derivatives at the LGR points leads to the notion of a differentiation matrix:

$$\hat{D}_{kj}[\alpha] = \frac{d}{d\tau} \left(\frac{\alpha(\tau)}{\alpha(\tau_j)} \phi_j(\tau) \right) \Bigg|_{\tau = \tau_k} \tag{10}$$

For $\alpha(\tau) = 1$, we denote $\hat{D}_{kj}[\alpha]$ as simply $\hat{D}_{kj}$; this is given explicitly by [31]

$$\hat{D}_{kj} := \begin{cases} \frac{-N(N+2)}{4} & j = k = 0, \\ \frac{L_N(\tau_k)}{L_N(\tau_j)} \frac{1-\tau_j}{1-\tau_k} \frac{1}{\tau_k - \tau_j} & j \neq k, \quad 1 \leq j, \quad k \leq N \\ \frac{1}{2(1-\tau_k)} & 1 \leq j = k \leq N \end{cases} \tag{11}$$

For $\alpha(\tau) = 1 - \tau$, we denote $\hat{D}_{kj}[\alpha]$ as $\hat{D}^*_{kj}$. It is straightforward to show that the entries of this differentiation matrix are given by

$$\hat{D}^*_{kj} := \begin{cases} -\frac{(N+1)^2+1}{4} & j = k = 0, \\ \frac{L_N(\tau_k)}{L_N(\tau_j)} \frac{1}{\tau_k - \tau_j} & j \neq k, \quad 1 \leq j, \quad k \leq N, \\ \frac{-1}{2(1-\tau_k)} & 1 \leq j = k \leq N \end{cases} \tag{12}$$

C. Transformations to the Physical Domain

The physical domain is $[0, \infty)$, and the computational domain is $[-1, 1)$. The closure of the computational domain is exactly the same as the domain of the transformed problems B and B^λ. Consider now a function $[0, \infty) \ni t \mapsto x(t) \in \mathbb{R}$, which need not be a state variable. Following the approximation theory described in Sec. IV.B, the Nth-order approximation of the function, $x^N(t)$, is given by

$$x^N(t(\tau)) = \sum_{j=0}^{N} \frac{\alpha(\tau)}{\alpha(\tau_j)} \phi_j(\tau) x_j \tag{13}$$

Thus,

$$x_j := x^N(t(\tau_j)) = x^N(t_j) \tag{14}$$

where

$$t_j := t(\tau_j) \in [0, \infty)$$

are called the shifted LGR points. It is clear that $t_j \neq \infty$ for any $j \in \{0, 1, \ldots, N\}$, but that $t_N \to \infty$ as $N \to \infty$. From Eqs. (4) and (6), it follows that

$$\int_0^\infty x(t) dt = \int_{-1}^{1} x(t(\tau)) r(\tau) d\tau \approx \sum_{j=0}^{N} w_j x_j \tag{15}$$

where

$$w_j := r(\tau_j) \hat{w}_j \tag{16}$$

are the quadrature weights in the physical domain. Similarly, we can derive an expression for the derivative of $x^N(t)$ at the shifted LGR points, t_k, using the chain rule:

$$\left.\frac{dx^N(t)}{dt}\right|_{t=t_k} = \frac{dx^N(t(\tau))}{d\tau}\left.\frac{d\tau}{dt}\right|_{\tau=\tau_k} \tag{17}$$

Let M be a diagonal matrix with entries

$$M_{kk} = \left.\frac{d\tau}{dt}\right|_{\tau=\tau_k} = \frac{1}{r(\tau_k)} \tag{18}$$

Then, following Eq. (10), we can write

$$\dot{x}^N(t_k) = \sum_{j=0}^{N} D_{kj}[\alpha] x_j \tag{19}$$

where

$$D_{kj}[\alpha] := M_{kk}\hat{D}_{kj}[\alpha] \tag{20}$$

D. Radau Advantage

As noted earlier, one of the reasons we used Radau points for the Legendre PS method is the singularity in $r(\tau)$ at $\tau = 1$. By avoiding this point, we avoid the singularity. The choice of Radau points is merely a pseudospectral approach for handling this singularity as opposed to ignoring it. Nonetheless, Radau points offer an advantage for PS methods for control that are not offered by Lobatto points. A distribution of the Lobatto and Radau points over the physical domain $[0, \infty)$ is shown in Fig. 1 for $N = 15$. Although both the Lobatto and Radau points cluster at the initial time, the Radau points are further spread out over the time axis at instances well beyond the initial time, toward $t \to \infty$. This implies that although both the Lobatto and Radau points offer good anti-aliasing properties [32] at the initial time, the Radau points provide a more accurate representation of the infinite horizon by distributing more points toward $t \to \infty$ (see Fig. 1). This implies a better representation of controls for feedback in terms of generating Carathéodory-π-type feedback solutions [14,15].

V. Solving Problem ${}^{\infty}B$

As discussed in detail in [12,29,33], the general theory of pseudospectral methods for optimal control relies on generating a consistent approximation for problem ${}^{\infty}B$, the main problem described in Sec. II. That is, the approximation scheme for problem ${}^{\infty}B$ must be consistent with problem ${}^{\infty}B^{\lambda}$. This notion of consistent approximations is not limited to PS methods alone; rather, this type of consistency is required for convergence of approximations for Runge–Kutta methods [34,35], including the popular Hermite–Simpson method. This notion of consistent approximations is different from those of Polak [36] and hence must be considered as additional requirements in generating convergent approximations [33]. The totality of these new concepts is encapsulated in the covector mapping principle [12,33,34,37–39] (CMP), as depicted in Fig. 2. In this figure, the prefix ∞ may be used to denote the problems discussed in Sec. II. Note also that problem B^{λ}, introduced in Sec. III, is represented in Fig. 2 as a "Pontryagin lift" of problem B. Because problems B and B^{λ} are equivalent to problems ${}^{\infty}B$ and ${}^{\infty}B^{\lambda}$, respectively, we simply need to apply the CMP to the transformed finite-horizon problems. That is, we first need to generate a CMP-consistent set of approximations for problems B and B^{λ}.

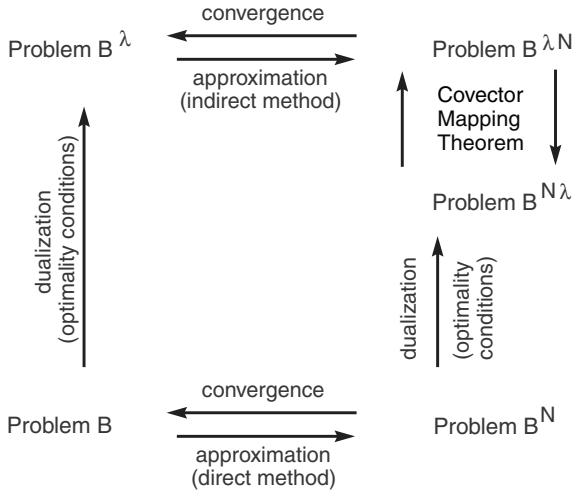

Fig. 2 Illustrating the covector mapping principle as first promulgated by Ross and Fahroo [9,37,44].

A. Generating Problems B^N and $B^{\lambda N}$

An application of the CMP for PS methods requires that we use the following approximations for the states and costates:

$$\boldsymbol{x}^N(t(\tau)) = \sum_{j=0}^{N} \phi_j(\tau)\boldsymbol{x}_j \tag{21}$$

$$\boldsymbol{\lambda}^N(t(\tau)) = \sum_{j=0}^{N}\left(\frac{1-\tau}{1-\tau_j}\right)\phi_j(\tau)\boldsymbol{\lambda}_j \tag{22}$$

In other words, to generate a CMP-consistent set of approximations, we need to choose different weights for interpolants that approximate the states and costates; that is, we require that $\alpha(\tau) = 1$ for the states and $\alpha(\tau) = 1 - \tau$ for the costates. Additional discussions of this notion specific to pseudospectral methods are discussed in [29]. For an introduction to the general principles, see [33]; for advanced topics, see [38]. As a simple illustration of the principle behind Eq. (22), observe that a substitution of $\tau = 1$ guarantees the transversality condition $\boldsymbol{\lambda}(t(1)) = \boldsymbol{0}$ in problem B^{λ}, but $\tau = 1$ is not part of the set of LGR points $\{\tau_j\}_{j=0}^{N}$ for any finite N. Because $\tau_N \to 1$ as $N \to \infty$, it is clear that

$$\lim_{t\to\infty}\boldsymbol{\lambda}^N(t) \to \boldsymbol{0} \quad \text{as } N \to \infty$$

in exactly the manner stipulated for problem ${}^{\infty}B^{\lambda}$ in Sec. II. Thus, applying the results of Sec. IV to problems B, we generate problem B^N:

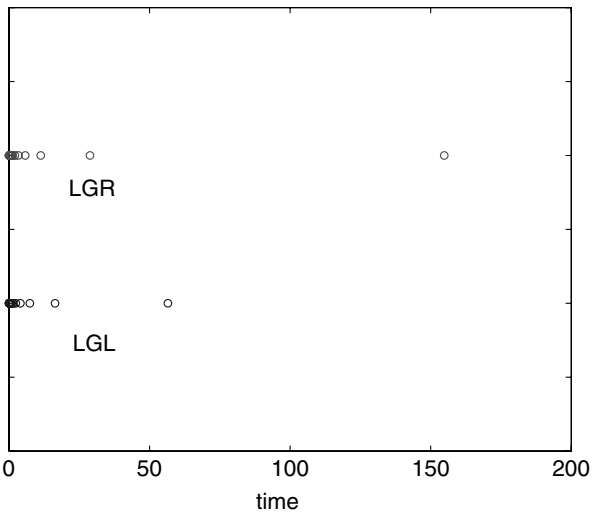

Fig. 1 Sample distribution of shifted Radau and Lobatto points.

$$[\boldsymbol{x}_k] := (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N), \qquad [\boldsymbol{u}_k] := (\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_N),$$

$$(B^N)\begin{cases} \text{Minimize} & J^N([\boldsymbol{x}_k], [\boldsymbol{u}_k]) = \sum_{k=0}^N F(\mathbf{x}_k, \mathbf{u}_k) w_k \\ \text{Subject to} & \sum_{j=0}^N D_{kj} \boldsymbol{x}_j - \mathbf{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) = \mathbf{0} \\ & \boldsymbol{x}_0 - \boldsymbol{x}^0 = \mathbf{0} \\ & \mathbf{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) \le \mathbf{0} \\ & k = 0, \ldots, N \end{cases}$$

where D_{kj} is $D_{kj}[\alpha]$ with $\alpha(\tau) = 1$. Similarly, using the notation D^*_{kj} for $D_{kj}[\alpha]$ with $\alpha(\tau) = 1 - \tau$, an approximation to problem B^λ is given by problem $B^{\lambda N}$, as given next:

$$(B^{\lambda N})\begin{cases} \text{Find} & ([\boldsymbol{x}_k], [\boldsymbol{u}_k], [\boldsymbol{\lambda}_k], [\boldsymbol{\mu}_k], \boldsymbol{\nu}_0) \\ \text{Such that} & \sum_{j=0}^N D_{kj} \boldsymbol{x}_j - \mathbf{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) = \mathbf{0} \\ & \boldsymbol{x}_0 - \boldsymbol{x}^0 = \mathbf{0} \\ & \mathbf{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) \le \mathbf{0} \\ & \frac{\partial(\bar{H}\boldsymbol{\mu}_k, \boldsymbol{\lambda}_k, \boldsymbol{x}_k, \boldsymbol{u}_k)}{\partial \boldsymbol{u}_k} = \mathbf{0} \\ & \boldsymbol{\mu}_k^T \mathbf{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) = \mathbf{0} \\ & \boldsymbol{\mu}_k \ge \mathbf{0} \\ & \sum_{j=0}^N D^*_{kj} \boldsymbol{\lambda}_j + \frac{\partial \bar{H}(\boldsymbol{\mu}_k, \boldsymbol{\lambda}_k, \boldsymbol{x}_k, \boldsymbol{\mu}_k)}{\partial \boldsymbol{x}_k} = \mathbf{0} \\ & \boldsymbol{\lambda}_0 = -\boldsymbol{\nu}_0 \\ & k = 0, \ldots, N \end{cases}$$

B. Remarks on the Transversality Conditions

Problem ${}^\infty B^\lambda$ (and, consequently, problem B^λ) is written with a terminal transversality condition (cf. Secs. II and III). An initial transversality condition of the form

$$\boldsymbol{\lambda}(t_0) = -\boldsymbol{\nu} \tag{23}$$

may also be included in the problem definition, but it is clearly redundant, because the initial transversality condition provides no new information toward constructing a candidate solution for problem B^λ. In fact, as argued by Pontryagin et al. [22], under appropriate conditions, problem B^λ contains the right number of boundary conditions to define a locally unique solution. An initial transversality condition is useful when some of the initial conditions are unknown. In this case, they provide the missing initial conditions (in dual space) to generate extremals. The discretized version of this problem (namely, problem $B^{\lambda N}$) is written with an initial transversality condition, but not the final transversality condition. The final transversality condition is automatically met by our choice of the interpolant for $t \mapsto \lambda(t)$ [cf. Eq. (22)]. Consequently, defining problem $B^{\lambda N}$ with a final transversality condition is redundant. In this context, our PS method has a Galerkin flavor. An initial transversality is introduced in problem $B^{\lambda N}$ because, unlike its continuous-time counterpart, we regard $\boldsymbol{x}_0$ as an unknown quantity. If we were to treat $\boldsymbol{x}_0$ to be given by $\boldsymbol{x}^0$, then the discretized form of the initial transversality condition

$$\boldsymbol{\lambda}_0 = -\boldsymbol{\nu}_0 \tag{24}$$

is unnecessary in the problem formulation (i.e., in problem $B^{\lambda N}$). In this case, with $\boldsymbol{x}_0$ being regarded as a known quantity, the index k for the unknown variables $\boldsymbol{x}_k$ would now run from $k = 1, \ldots, N$, whereas the corresponding index for the remaining variables would run from $k = 0$ to N. We prefer to use a uniform indexing scheme, in which case, we regard $\boldsymbol{x}_0$ as an "artificial" unknown variable constrained by $\boldsymbol{x}_0 - \boldsymbol{x}^0 = \mathbf{0}$. This preference for a uniform indexing scheme, consistent with treating $\boldsymbol{x}_0$ as an unknown variable, implies that the continuous-time initial transversality condition (23) must now be introduced as Eq. (24) in its discrete-time form, consistent with the notion of regarding $\boldsymbol{x}(t_0)$ as an unknown variable. Within the context of PS methods, this technique of "adding" equations to meet boundary conditions is known as the tau method [27]. Thus, it is clear that a PS method for optimal control has flavors of both the Galerkin and tau methods.

C. Developing Problem $B^{N\lambda}$

This problem is obtained by applying the multiplier theory for problem B^N. It is straightforward to show that by using $\{w_j\}_{j=0}^N$ to construct the discrete Lagrangian, we generate the following problem:

$$(B^{N\lambda})\begin{cases} \text{Find} & ([\boldsymbol{x}_k], [\boldsymbol{u}_k], [\tilde{\boldsymbol{\lambda}}_k], [\tilde{\boldsymbol{\mu}}_k], \tilde{\boldsymbol{\nu}}_0) \\ \text{Such that} & \sum_{j=0}^N D_{kj} \boldsymbol{x}_j - \mathbf{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) = \mathbf{0} \\ & \boldsymbol{x}_0 - \boldsymbol{x}^0 = \mathbf{0} \\ & \mathbf{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) \le \mathbf{0} \\ & \frac{\partial \bar{H}(\tilde{\boldsymbol{\mu}}_k, \tilde{\boldsymbol{\lambda}}_k, \boldsymbol{x}_k, \boldsymbol{u}_k)}{\partial \boldsymbol{u}_k} = \mathbf{0} \\ & \tilde{\boldsymbol{\mu}}_k^T \mathbf{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) = \mathbf{0} \\ & \tilde{\boldsymbol{\mu}}_k \ge \mathbf{0} \\ \text{for} & k = 0, \ldots, N \\ & \sum_{j=0}^N D^*_{kj} \tilde{\boldsymbol{\lambda}}_j + \frac{\partial \bar{H}(\tilde{\boldsymbol{\mu}}_k, \tilde{\boldsymbol{\lambda}}_k, \boldsymbol{x}_k, \boldsymbol{\mu}_k)}{\partial \boldsymbol{x}_k} = \mathbf{0} \\ \text{for} & k = 1, \ldots, N \\ \text{and} & \sum_{j=0}^N D^*_{0j} \tilde{\boldsymbol{\lambda}}_j + \frac{\partial \bar{H}(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\lambda}}_0, \boldsymbol{x}_0, \boldsymbol{\mu}_0)}{\partial \boldsymbol{x}_0} = -\mathbf{c}_0 \\ & \tilde{\boldsymbol{\lambda}}_0 + \tilde{\boldsymbol{\nu}}_0 = w_0 \mathbf{c}_0 \end{cases}$$

where $\mathbf{c}_0 \in \mathbb{R}^{N_x}$ is arbitrary.

D. Covector Mapping Theorem

When the multiplier theory is applied to problem B^N, the resulting Karush–Kuhn–Tucker (KKT) conditions are not automatically in the form of problem $B^{N\lambda}$. The equations need to be rearranged or transformed to render them similar to problem $B^{\lambda N}$. This transformation of the equations for PS methods is quite straightforward once the correct primal-dual interpolants are identified, as in Eqs. (21) and (22). A somewhat similar concept is necessary in generating discrete Runge–Kutta approximations, as observed by Hager [35]. As a matter of fact, if it does not satisfy Hager's conditions, a Runge–Kutta method that is convergent as an integrator for a differential equation will be divergent as a collocation method for optimal control. In any event, it is quite clear, by mere inspection, that any solution to problem $B^{\lambda N}$ is also a solution to problem $B^{N\lambda}$; in this case, we have

$$\mathbf{c}_0 = \mathbf{0} \tag{25}$$

That is, Eq. (25) is a matching condition that matches problem $B^{N\lambda}$ to problem $B^{\lambda N}$. Matching conditions are part of the totality of closure conditions required to complete the circuit (arrows) indicated in Fig. 2. *Note that problems B^N and $B^{\lambda N}$ are generated from problems B and B^λ, respectively, without introducing any additional continuous-time primal conditions to carry over to the discrete-time problems*. This notion is implicit in Fig. 2.

In completing the steps suggested in Fig. 2 toward the development of a covector mapping theorem, we identify the following multiplier sets analogous to those introduced in [40]: Let $\chi := \{[\boldsymbol{x}_k], [\boldsymbol{u}_k]\}$ and $\Lambda := \{\boldsymbol{\nu}_0, [\boldsymbol{\mu}_k], [\boldsymbol{\lambda}_k]\}$. We denote by $\mathbb{M}^{\lambda N}(\chi)$ the multiplier set corresponding to χ,

$$\mathbb{M}^{\lambda N}(\chi) := \{\Lambda : \Lambda \text{ satisfies the conditions of problem } B^{\lambda N}\} \tag{26}$$

Similarly, we define $\tilde{\Lambda} := \{\tilde{\boldsymbol{\nu}}_0, [\tilde{\boldsymbol{\mu}}_k], [\tilde{\boldsymbol{\lambda}}_k]\}$ and $M^{N\lambda}(\chi)$, the multiplier set:

$$\mathbb{M}^{N\lambda}(\chi) := \{\tilde{\Lambda} : \tilde{\Lambda} \text{ satisfies the conditions of problem } B^{N\lambda}\} \tag{27}$$

Clearly, $\mathbb{M}^{\lambda N}(\chi) \subseteq \mathbb{M}^{N\lambda}(\chi)$. We now define a new multiplier set:

$$\hat{\mathbb{M}}^{N\lambda}(\chi) := \{\tilde{\Lambda} \in \mathbb{M}^{N\lambda}(\chi) : \tilde{\Lambda} \text{ satisfies } \mathbf{c}_0 = \mathbf{0}\} \tag{28}$$

Thus, $\hat{\mathbb{M}}^{N\lambda}(\chi) \sim \mathbb{M}^{\lambda N}(\chi)$. That is, under a matching (closure) condition, every solution of problem $B^{N\lambda}$ is also a solution to problem $B^{\lambda N}$.

Theorem 1 (covector mapping theorem): Let $\mathbb{M}^{\lambda N}(\chi) \neq \emptyset$ and $\{\hat{\boldsymbol{\nu}}_0, [\hat{\boldsymbol{\mu}}_k], [\hat{\boldsymbol{\lambda}}_k]\} \in \hat{\mathbb{M}}^{N\lambda}(\chi)$, then the bijection $\hat{\mathbb{M}}^{N\lambda}(\chi) \sim \mathbb{M}^{\lambda N}(\chi)$ is given by

$$\boldsymbol{\lambda}^N(t_k) = \hat{\boldsymbol{\lambda}}_k, \qquad \boldsymbol{\mu}^N(t_k) = \hat{\boldsymbol{\mu}}_k, \qquad \boldsymbol{\nu}_0 = \hat{\boldsymbol{\nu}}_0 \tag{29}$$

Remark 1.1: Although the statement of Theorem 1 is identical to that of [40] for the Legendre–Gauss–Lobatto (LGL) method, note that the problems $B^{N\lambda}$ and $B^{\lambda N}$ for the Radau methods are defined differently through the construction of weighted interpolation. The weighted interpolants lead to a consistent pair of differentiation matrices D and D^* that are dual to each other. In this context, the LGL case turns out to be the special situation in which the formal adjoint of D is the same as $-D$.

Remark 1.2: Note that Theorem 1 neither states $\boldsymbol{\lambda}^N(t_k) = \tilde{\boldsymbol{\lambda}}_k$, $\boldsymbol{\mu}^N(t_k) = \tilde{\boldsymbol{\mu}}_k$, or $\boldsymbol{\nu}_0 = \tilde{\boldsymbol{\nu}}_0$, nor does it imply $\boldsymbol{\lambda}^N(t_k) \approx \tilde{\boldsymbol{\lambda}}_k$, $\boldsymbol{\mu}^N(t_k) \approx \tilde{\boldsymbol{\mu}}_k$, or $\boldsymbol{\nu}_0 \approx \tilde{\boldsymbol{\nu}}_0$, as is sometimes erroneously interpreted. Furthermore, note that Eq. (29) is an exact relationship for both the LGR and LGL PS methods.

E. Remarks on the Covector Mapping Principle

The covector mapping theorem was generated by an application of the covector mapping principle (see Fig. 2). Since the introduction of Pontryagin's principle, it has been known that the minimum principle may fail in the discrete-time domain if it is applied in exactly the same manner as in the continuous-time domain. For the minimum principle to hold exactly in the discrete-time domain, additional assumptions of convexity are required, whereas no such assumptions are necessary for the continuous-time versions. This is because continuous time has a convexity property hidden through its continuity. Thus, when a continuous-time optimal control problem is discretized, the time convexity is not carried over to the discrete-time domain, resulting in a loss of information. If this information loss is not restored, the discrete-time solution may be spurious, not converge to the correct solution, or may even provide completely false results. A historical account of these issues, along with a simple counter example, is described in [37]. Further details are provided in the references contained in [37]. A thorough discussion of these issues and their relationship to advance concepts in optimal control theory has been developed by Mordukhovich [38,39].

The closure conditions introduced by Ross and Fahroo [12] are a form of matching conditions that are similar in spirit to Mordukhovich's [38,39] matching conditions for Euler approximations. These conditions can be in primal space alone [35,38] or in primal–dual space [33]. Note, however, that the primal-space conditions [35,38] are obtained through dual-space considerations. These new ideas reveal that dual-space issues cannot be ignored even in the so-called direct methods for optimal control.

F. Software

A proper implementation of a PS method requires addressing all the numerical stability and accuracy issues, and a case-by-case approach to problem-solving using PS approximations is inadvisable because it is tantamount to using first principles in every situation. Consequently, it is preferable for all the intricacies of a PS method to be implemented just once for a general problem in a form of a reusable software package. DIDO [41] is such a general-purpose application package for solving optimal control problems within the MATLAB® environment. It is a minimalist's approach to solving optimal control problems, in that only the problem formulation is required in a manner akin to writing it on a piece of paper. DIDO automatically completes the "circuit" shown in Fig. 2. The Radau approach is implemented in an α version of the software package. This software was used in the following sections that illustrate the principles and practice.

VI. Example 1: Linear-Quadratic Regulator

We first consider an LQR problem to illustrate and validate the LGR approach vis-a-vis the Riccati technique. Thus, this example is not to be construed as illustrating a replacement of Riccati techniques. To this end, consider the following LQR problem from Kirk [42]. The problem is to find the optimal state-control function pair that minimize the quadratic cost functional

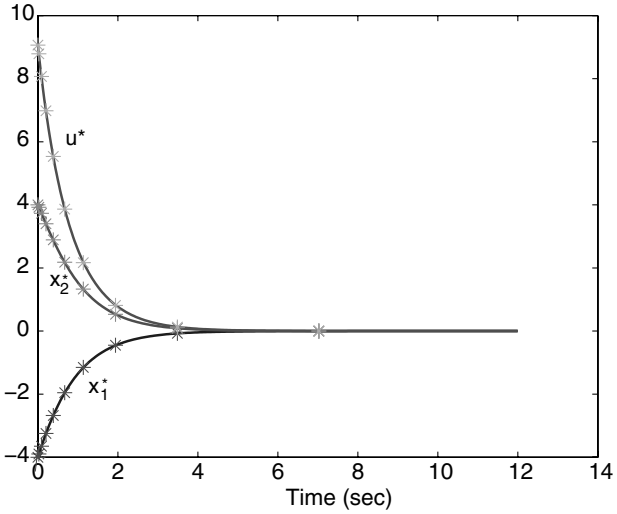

Fig. 3 Sample LQR results for the states and control using LGR PS (stars) and Riccati techniques (solid line).

$$J[x(\cdot), u(\cdot)] = \int_0^\infty \left(x_1^2(t) + \frac{1}{2}x_2^2(t) + \frac{1}{4}u^2(t) \right) dt \tag{30}$$

subject to the second-order dynamics

$$\dot{x}_1 = x_2(t) \tag{31}$$

$$\dot{x}_2 = 2x_1(t) - x_2(t) + u(t) \tag{32}$$

and initial condition $[x_1(0), x_2(0)] = [-4, 4]$. The standard approach for solving this problem is to numerically solve the algebraic Riccati equation for constructing a gain matrix and to use the resulting optimal feedback control law to integrate the state equations forward for a sufficiently long horizon to obtain the optimal states. The optimal control computed from the matrix Riccati solution via the gain matrix $\mathbf{K}$ is given by [42]

$$u^* = -2[K_1 \;\; K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{33}$$

where K_1 and K_2 are the components of the matrix Riccati solution. The resulting states and controls from both the Riccati approach and the Radau PS technique are shown in Fig. 3. The solid line represents the Riccati-based solution, and the stars denote the Radau PS solutions for $N = 12$. Evidently, our method can accurately solve the LQR problem for a small number of nodes.

The plots in Figs. 4 and 5 show the costates obtained from the Riccati solution (solid lines):

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = K\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{34}$$

The stars in these plots are the costates obtained from Theorem 1. From these plots, it is obvious that the costates obtained from the covector mapping theorem are at least as accurate at the Riccati approach.

VII. Example 2: Attitude Hold for NPSAT1

NPSAT1 (see Fig. 6) is an experimental spacecraft that was conceived, designed, and built at the Naval Postgraduate School (NPS) and is scheduled to be launched in fall 2009. This spacecraft has formed a testbed at NPS for illustrating various new ideas in constrained nonlinear control and estimation [15,43]. To illustrate

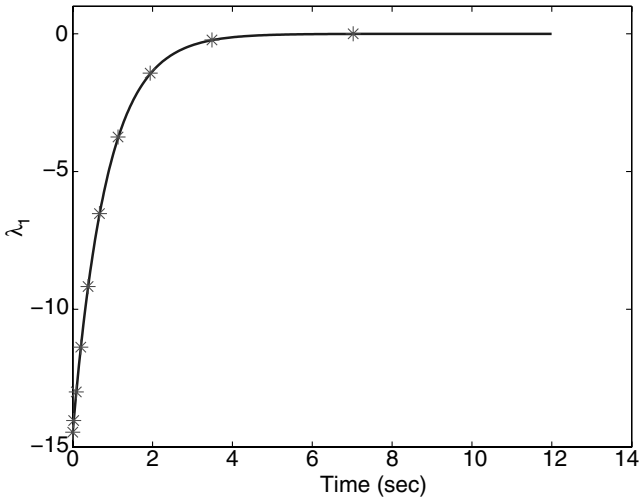

Fig. 4 Comparison of the results for $\lambda_1(t)$ using LGR PS (stars) and Riccati techniques (solid line).

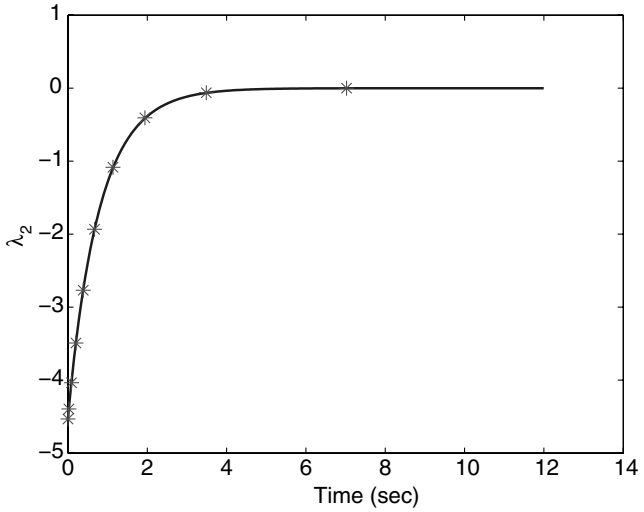

Fig. 5 Comparison of the results for $\lambda_2(t)$ using LGR PS (stars) and Riccati techniques (solid line).

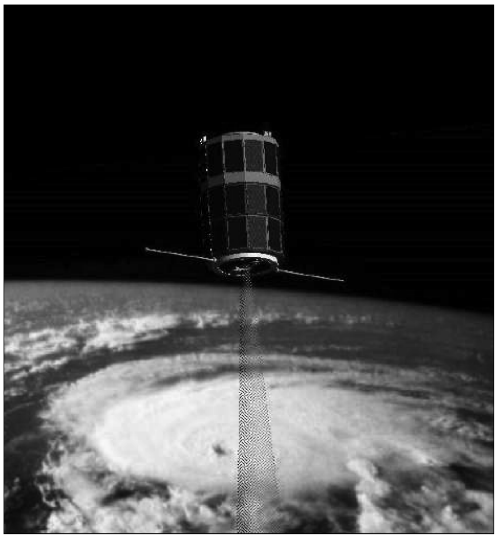

Fig. 6 Artist's rendition of NPSAT1 in orbit.

the key ideas of the LGR method, we use a standard rigid-body dynamic model for NPSAT1:

$$\dot{q}_1(t) = \frac{1}{2}[\omega_x(t)q_4(t) - \omega_y(t)q_3(t) + \omega_z(t)q_2(t)]$$

$$\dot{q}_2(t) = \frac{1}{2}[\omega_x(t)q_3(t) + \omega_y(t)q_4(t) - \omega_z(t)q_1(t)]$$

$$\dot{q}_3(t) = \frac{1}{2}[-\omega_x(t)q_2(t) + \omega_y(t)q_1(t) + \omega_z(t)q_4(t)]$$

$$\dot{q}_4(t) = \frac{1}{2}[-\omega_x(t)q_1(t) - \omega_y(t)q_2(t) - \omega_z(t)q_3(t)]$$

$$\dot{\omega}_x(t) = \frac{I_2 - I_3}{I_1}\omega_y(t)\omega_z(t) + u_1$$

$$\dot{\omega}_y(t) = \frac{I_3 - I_1}{I_2}\omega_x(t)\omega_z(t) + u_2$$

$$\dot{\omega}_z(t) = \frac{I_1 - I_2}{I_3}\omega_x(t)\omega_y(t) + u_3$$

where q_i $(i = 1, 2, 3, 4)$ are the quaternion parameters that satisfy the state constraints

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{35}$$

$$q_4 \geq 0 \tag{36}$$

u_i $(i = 1, 2, 3)$ are the control torques that satisfy the control constraints

$$|u|_i \leq 0.01\ N \cdot m \qquad i = 1, 2, 3 \tag{37}$$

and $I_1 = 5$, $I_2 = 5.1$, and $I_3 = 2$ are the inertias of NPSAT1. The initial condition corresponding to a rest position at 30 deg off each axis is given by

$$[q_1, q_2, q_3, q_4] = [0.91856, 0.17678, 0.30619, 0.17678] \tag{38}$$

$$[\omega_x, \omega_y, \omega_z] = [0, 0, 0] \tag{39}$$

The goal is to regulate NPSAT1 to its equilibrium position at

$$[q_1, q_2, q_3, q_4] = [0, 0, 0, 1], \qquad [\omega_x, \omega_y, \omega_z] = [0, 0, 0]$$

To solve this problem, we construct a quadratic cost function:

$$F(e, \omega, u) = \|e\|^2 + 10\|\omega\|^2 + 100\|u\|^2$$

where $e := [e_1, e_2, e_3, e_4]$ is the error quaternion

$$[e_1, e_2, e_3, e_4] = [q_1, q_2, q_3, q_4 - 1]$$

and $\omega := [\omega_x, \omega_y, \omega_z]$ and $u := [u_1, u_2, u_3]$.

The LGR PS control solution for $N = 60$ is shown in Fig. 7. Note that $t_l \to u_1$ and $t_l \to u_3$ are saturated early on. The corresponding state trajectories are shown in Figs. 8 and 9. When the initial condition [see Eqs. (38) and (39)] is propagated through the dynamics with the interpolated PS controls, the resulting trajectories are indeed feasible, as illustrated in Figs. 8 and 9. Having established the feasibility of the PS controls, we now test their optimality. The control Hamiltonian is given by

$$H(\lambda, q, \omega, u) = q_1^2 + q_2^2 + q_3^2 + (q_4 - 1)^2 + 10\left(\omega_x^2 + \omega_y^2 + \omega_z^2\right) + 100\left(u_1^2 + u_2^2 + u_3^2\right) + \lambda_5 u_1 + \lambda_6 u_2 + \lambda_7 u_3 + H_0$$

where H_0 denotes terms in the Hamiltonian function that are not dependent upon u. An application of the KKT conditions for the Hamiltonian minimization condition,

$$\begin{cases} \text{Minimize}_u & H(\lambda, q, \omega, u) \\ \text{Subject to} & \|u_i\| \leq 0.01, i = 1, 2, 3 \end{cases} \tag{40}$$

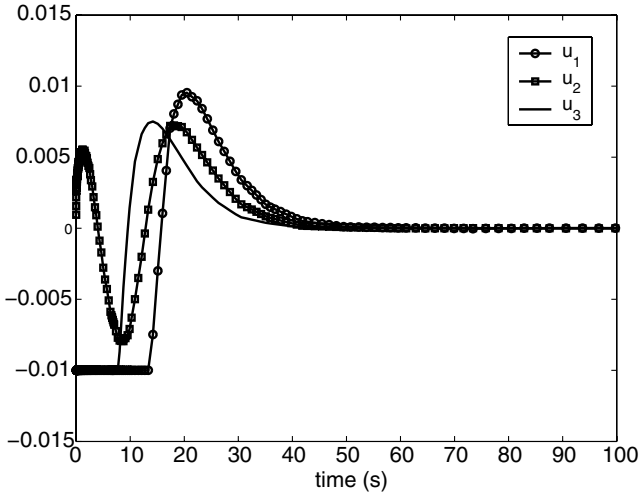

Fig. 7 Candidate LGR/PS-optimal control for stabilizing NPSAT1.

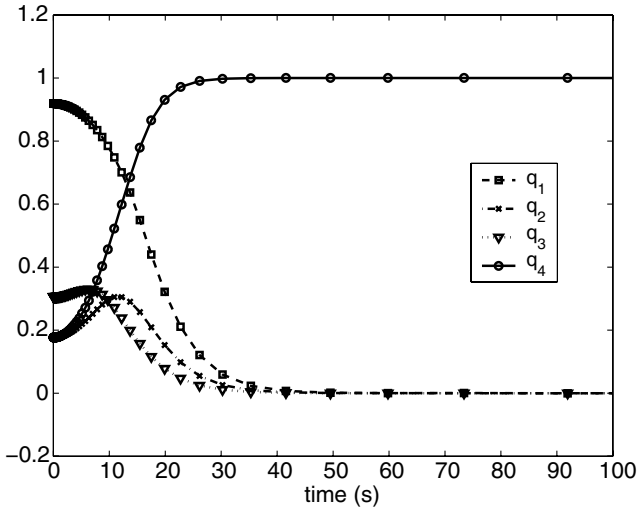

Fig. 8 Quaternions for the NPSAT1 stability problem; stars represents the discrete solution with 60 nodes and the solid lines are propagated trajectories.

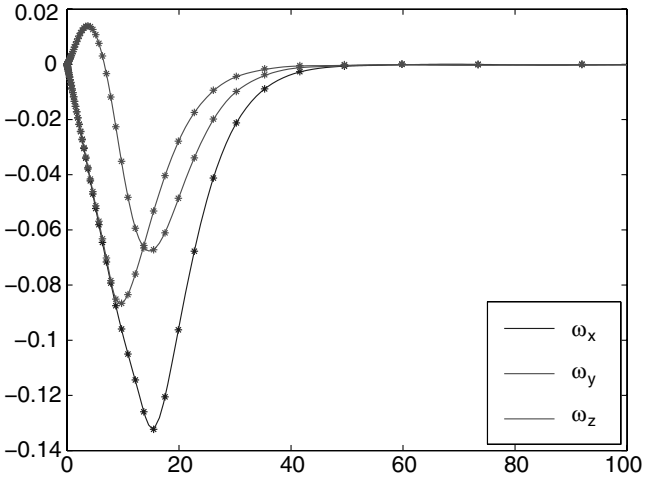

Fig. 9 Angular velocities for the NPSAT1 stability problem; stars represents the discrete solution with 60 nodes and the solid lines are the propagated trajectories.

requires that

$$u_i = \begin{cases} 0.01 & \text{if } -\lambda_{i+4}/200 > 0.01 \\ -0.01 & \text{if } -\lambda_{i+4}/200 < 0.01 \\ -\lambda_{i+4}/200 & \text{otherwise} \end{cases} \tag{41}$$

The costates obtained from an application of Theorem 1 are plotted alongside the controls in Figs. 10–12. It is quite obvious from these figures that Eq. (41) is satisfied at all points, and hence we conclude

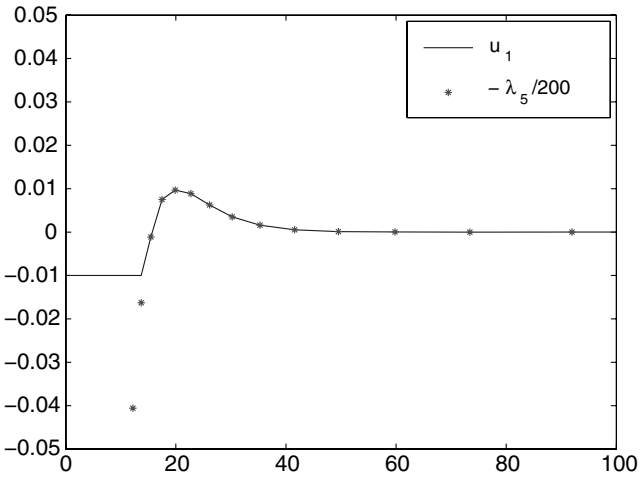

Fig. 10 Control trajectory $t \mapsto u_1$ and its switching function.

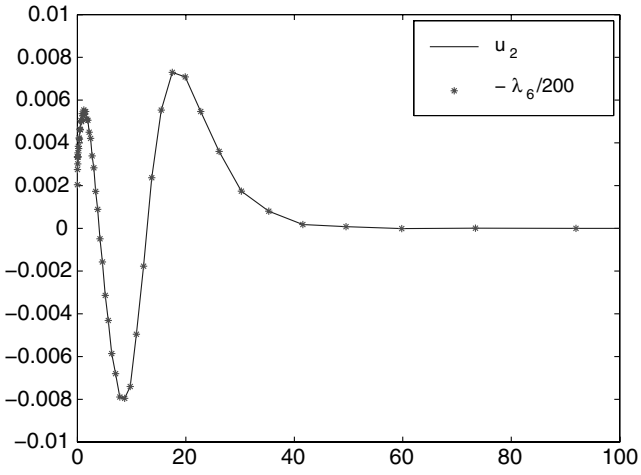

Fig. 11 Control trajectory $t \mapsto u_2$ and its switching function.

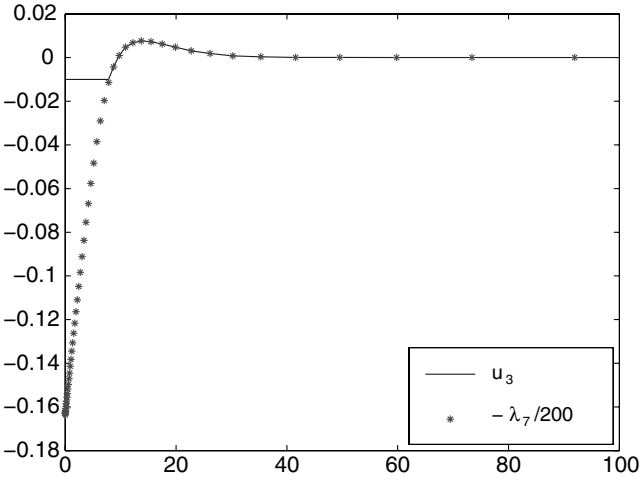

Fig. 12 Control trajectory $t \mapsto u_3$ and its switching function.

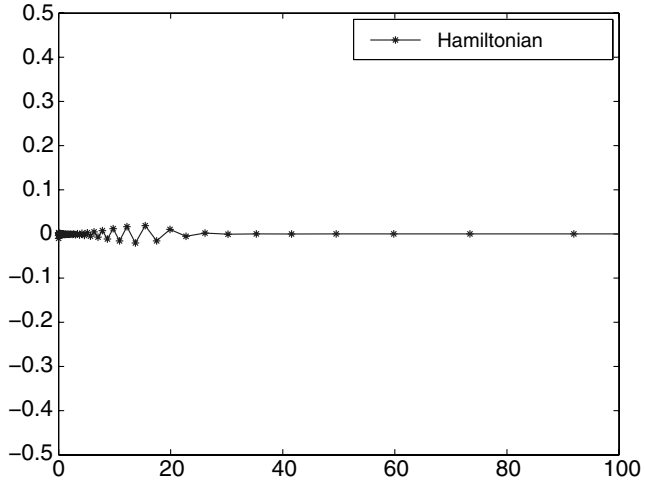

Fig. 13 Evolution of the minimized Hamiltonian.

that the LGR PS controls satisfy the Hamiltonian minimization condition.

The Hamiltonian value condition requires that $H[t_f] = 0$, whereas the Hamiltonian evolution equation implies that $t \mapsto h$ is a constant. A construction of the Hamiltonian based on Theorem 1 is plotted in Fig. 13. From this plot, it is clear that both the Hamiltonian value condition and the Hamiltonian evolution equation are satisfied within numerical precision. Hence, it can be argued that the LGR PS controls are at least Pontryagin extremals.

VIII. Additional Examples, Computational Speed, Etc.

As noted in Sec. I, many more example problems have been solved in [7,14]. In particular, the inverted pendulum problem, with all its nonlinearities and saturation constraints, has been considered in [7,14]. In nearly all these problems, solutions were obtained without implementing any of the computational speed enhancements discussed elsewhere [3]. Despite this rudimentary implementation, the run times to solve these problems on a Pentium 4 processor range from fractions of a second to a few seconds. For instance, in [7], the nonlinear inverted pendulum problem was solved in about 2 s. When some (not all) of the computational enhancements suggested in [3,32] were implemented [14], an average run time of 0.08 s was obtained. The average was based on over 100 runs with randomly chosen initial conditions. This exercise not only illustrates the possible enhancements in computational speed by algorithmic means alone, but also the robustness of the approach. In [14], we have also demonstrated that this run time is more than adequate to generate feedback solutions. For linear-quadratic problems, Yan et al. [20] showed that it is possible to generate a closed-form solution using PS methods and that this solution can be generated by an order of magnitude faster than the solution obtained by the Riccati equation. A full discussion on using PS methods for feedback stabilization is beyond the scope of this paper; hence, we simply refer to [14] for details that include a design and proof of stabilization by feedback control, with the PS method constituting the computational technique. We also refer the reader to [15] for further details that include an experimental demonstration. In any event, what has emerged in recent years is that true real-time solutions are fully realizable with modern techniques without any expectation of further improvements in computer hardware.

IX. Conclusions

Receding-horizon techniques for solving infinite-horizon optimal control problems introduce many new problems arising chiefly due to the mismatch between the computational horizon and the infinite horizon. A nonreceding Legendre–Gauss–Radau (LGR) pseudospectral (PS) method is developed and proposed as a viable method to solve infinite-horizon, nonlinear, constrained optimal control problems. An application of the covector mapping principle for the LGR PS methods produces a covector mapping theorem that allows one to generate dual maps without resorting to solving difficult boundary-value problems. Numerical experiments with the LGR PS method indicate spectral convergence. In this regard, this paper provides the computational foundations for a new approach to stabilizing constrained nonlinear control systems. Consequently, it has opened up a large number of new avenues for research in nonlinear control. These topics are at the intersection of control, optimization, approximation theory, and computational mathematics. Given that our proposed method appears to be quite powerful, the new avenues of research it promises appear to be quite exciting.

Acknowledgments

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